87-32

Szigorlat

(Analízis)

## Infimum, Supremum, Min., Max.

1, $H := \left\{ \frac{3n-1}{2n+5} \mid n \in \mathbb{N} \right\}$ 		zralatos

$\left\{ \frac{2}{7}; \frac{5}{9}; \frac{8}{11}; \frac{11}{13}; \dots \right\}$

$\min(H) = \frac{2}{7} = \inf(H)$

$\max(H) = \emptyset$

2, $H := \left\{ \frac{k}{n} \mid k, n \in \mathbb{N},\ n \geq 2,\ 1 \leq k \leq n-1 \right\}$

$\left\{ \frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \frac{2}{3}; \frac{2}{4}; \frac{3}{4}; \dots \frac{99}{100}; \dots \frac{399}{400} \right\}$

$\min(H) = \emptyset$ 	$\max(H) = \emptyset$ 		zralatos

$\sup(H) = 1$ 	$\inf(H) = 0$

3, $H := (-1, 1) \cap \mathbb{Q}$

$\inf(H) = -1$ 	$\sup(H) = 1$

$\min, \max = \emptyset$

## Számtani, Mértani közép

1, Igazoljuk szt. és mt közép alkalmazásával

$\forall n \in \mathbb{N}$

$\forall a_1, \dots, a_n > 0$

$$\bar{x} = \frac{1}{n} \sum_{k=1}^{n} a_k \geq \sqrt[n]{\prod_{k=1}^{n} a_k}$$

$$\underbrace{(a_1 + \dots + a_n)}_{a} \cdot \underbrace{\left(\frac{1}{a_1} + \dots + \frac{1}{a_n}\right)}_{b} \geq n^2$$

a, $\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}$

b,

$$\frac{\frac{1}{a_1}+\dots+\frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1}\cdot\frac{1}{a_2}\cdot\dots\cdot\frac{1}{a_n}}$$

$$\frac{(a_1+\dots+a_n)\left(\frac{1}{a_1}+\dots+\frac{1}{a_n}\right)}{n^2} \geq \sqrt[n]{\underbrace{(a_1\cdot\dots\cdot a_n)\left(\frac{1}{a_1}\cdot\dots\cdot\frac{1}{a_n}\right)}_{1}}$$

$$\sqrt[n]{1}=1$$

$$(a_1+\dots+a_n)\left(\frac{1}{a_1}+\dots+\frac{1}{a_n}\right) \geq n^2$$

2, Adott $k>0$ kerületű téglalapok közül melyek a legnagyobb területűek?

$$K=2(a+b) \qquad T=a\cdot b \qquad \frac{a+b}{2}\geq\sqrt{ab}$$

$$\frac{K/2}{2}\geq\sqrt{T}$$

$$\frac{K}{4}\geq\sqrt{T}$$

T akkor a legnagyobb, ha $\sqrt{T}=\frac{K}{4}$

$\Rightarrow$ csak akkor, ha $a=b$

b, $a=\frac{K}{2}-b$ $\qquad$ $T=a\cdot b$ maximális!

$$T=\left(\frac{K}{2}-b\right)\cdot b=\frac{Kb}{2}-b^2$$

$$T'_b=\frac{K}{2}-2b \qquad \frac{K}{2}-2b=0$$

$$\frac{K}{2}=2b$$

$$2b=a+b$$

$$b=a \qquad a=b$$

# Teljes indukció

1, $2^n > n^2$ , ha $n > 4$ $n \in \mathbb{N}$

(1) $n = 5$ -re

$$2^5 > 5^2 \quad \checkmark$$

(2) indukciós feltevés

Tfh $2^k > k^2$

(3) $n+1$ -re

$$2^{(n+1)} > (n+1)^2$$

$$2 \cdot 2^n > n^2 + 2n + 1$$

$$2 \cdot n^2 > n^2 + 2n + 1$$

$$n^2 > 2n + 1$$

$$n > 2 + \frac{1}{n}$$

$$5 > 2 + \frac{1}{5} \quad \checkmark$$

-> mivel $2^k > k^2$ esetén $\forall$-re teljesül

2, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(1) $n = 1$ $\quad 1 = \frac{1(1+1)}{2} = 1 \quad \checkmark$

(2) Tfh $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(3) $n+1$ -re

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+1+1)}{2}$$

$$\sum_{k=1}^{n} k + (n+1) = \frac{(n+1)(n+2)}{2}$$

Ind felt $\quad \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$

$$\frac{n^2+n}{2}+n+1=\frac{n^2+n+2n+2}{2}$$ ✓

3, $$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$$

① $n=1$-re

$$1=\frac{1(2)(3)}{6}=1$$ ✓

② Tfh

– 〃 –

③ $n+1$-re

$$\sum_{i=1}^{n+1} i^2=\frac{(n+1)(n+1+1)(2(n+1)+1)}{6}$$

$$\sum_{i=1}^{n} i^2+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

Ind. feltl.

$$(n+1)^2+\frac{n(n+1)(2n+1)}{6}=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$\cancel{\frac{2n^3+3n^2+n}{6}}+n^2+2n+1=\cancel{\frac{2n^3+6n^2+4n+3n^2+9n+6}{6}}$$

$$(n^2+3n+2)(2n+3)$$

4, $4^n+15n-1 \quad /9$ ✓

– – –

③ $$4^{(n+1)}+15(n+1)-1=4\cdot 4^n+15n+15-1=$$

$$(4^n+15n-1)+3\cdot 4^n+15$$

✓ $$3\cdot 4^n+15 \quad /9$$ osztható, tehát négyszeres

$$3\cdot 4^{(n+1)}+15 \quad 3\cdot 4^n$$ osztható ✓

Döntse el a sorozatoknál, hogy konvergens vagy divergens, + határértékek

1, $\lim_{n\to\infty} \frac{n^2+3n-1}{n^3-7n^2+6n-10} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{3}{n^2}-\frac{1}{n^3}}{1-\frac{7}{n}+\frac{6}{n^2}-\frac{10}{n^3}} = \frac{0}{1} = \underline{\underline{0}}$

~~div~~ konvergens

2, $\lim_{n\to\infty} 3n^8+2n^5-7n+1 = \lim_{n\to\infty} \underset{\downarrow \atop \infty}{n^8}\left(3+\underset{\downarrow \atop 0}{\frac{2}{n^3}}-\underset{0}{\frac{7}{n^7}}+\underset{0}{\frac{1}{n^8}}\right) = +\infty$

div.

3, $\lim_{n\to\infty} \frac{\frac{1}{3}n+\frac{1}{2}}{-6n-1} = \lim_{n\to\infty} \frac{n}{n}\left(\frac{\frac{1}{3}+\frac{1}{2}/n}{-6-\frac{1}{n}}\right) = \frac{\frac{1}{3}}{-6} = \underline{\underline{-\frac{1}{18}}}$

konv.

4, $\lim_{n\to\infty} \left(n+3-\sqrt{n^2+2n-1}\right) = \lim_{n\to\infty} n\left(1+\frac{3}{n}-\sqrt{1+\frac{2}{n}-\frac{1}{n^2}}\right) =$

$\infty\,(1+0-\overbrace{\quad}^{}1)$

$= \underline{\underline{0}}$ konv.

5, $\lim_{n\to\infty} \frac{3n+\sqrt{n}}{5\sqrt{n}+\sqrt{n^2+8}} = \lim_{n\to\infty} \underset{\downarrow \atop 1}{\frac{n}{n}} \frac{3+\sqrt{\frac{1}{n}}^{\nearrow 0}}{5\underset{\downarrow \atop 0}{\sqrt{\frac{1}{n}}}+\underset{\downarrow \atop 1}{\sqrt{1+\frac{8}{n^2}}}} = \underline{\underline{3}}$

Keressünk küszöbindexet

$a_n = 1-10^{-n}, \quad \varepsilon = 0{,}001$

$|a_n - A| < \varepsilon$

$\lim_{n\to\infty} 1-\underset{\downarrow \atop 0}{10^{-n}} = \underline{\underline{1}}$

$|1-10^{-n}-1| < 10^{-3}$

$10^{-n} < 10^{-3}$ → $10^x$ fv. szig. mon. miatt

$-n < -3$

$n > 3$

$\underline{\underline{N = 3}}$

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2, $a_n = \frac{2}{(n+1)^2}$ , $\varepsilon = 10^{-4}$

$$\lim_{n\to\infty} \frac{2}{n^2+2n+1} = \lim_{n\to\infty} \frac{\frac{2}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} = \underline{\underline{0}}$$

$$\left|\frac{2}{(n+1)^2} - 0\right| < \frac{1}{10^4}$$

$$2\cdot 10^4 < (n+1)^2$$

$$\sqrt{2}\cdot 100 < n+1$$

$$100\sqrt{2} - 1 < n$$

$$140{,}42 < n$$

$$\underline{\underline{N = 141}}$$

3, $a_n = \frac{n-1}{2n+1}$ , $\varepsilon = 10^{-5}$

$$\lim_{n\to\infty} \frac{n-1}{2n+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2}$$

$$\left|\frac{n-1}{2n+1} - \frac{1}{2}\right| < 10^{-5}$$

$$n-1 < \left(10^{-5} + \frac{1}{2}\right)(2n+1)$$

$$n-1 < 1{,}00002n + 0{,}50001$$

$$n < 1{,}00002n + 1{,}50001$$

$$-0{,}00002 \cdot n < 1{,}50001$$

$$n > 74999{,}5 \qquad N = 7500$$

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## 'e'-hez tartó határértékek

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

1, $$\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)^n =$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1-\frac{1}{n}\right)^n} = \frac{e}{e^{-1}} = \frac{e}{\frac{1}{e}} = \underline{\underline{e^2}}$$

2, $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n+3} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}\cdot\left(1+\frac{1}{n}\right)^3 = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^2\cdot$$

$$\cdot\left(1+\frac{1}{n}\right)^3 = e^2\cdot 1 = \underline{\underline{e^2}}$$

3, $$\lim_{n\to\infty}\left(\frac{n-4}{n+5}\right)^{4n+1} = \lim_{n\to\infty}\left(\frac{n+5-9}{n+5}\right)^{4n+1} = \lim_{n\to\infty}\left(1-\frac{9}{(n+5)}\right)^{4n+1} =$$

$$= \lim_{n\to\infty}\underbrace{\left(1-\frac{9}{n+5}\right)^1}_{1}\cdot\left(\left(1-\frac{9}{n+5}\right)^n\right)^4 = \lim_{n\to\infty}\left(\left(1-\frac{9}{n+5}\right)^{n+5}\cdot\underbrace{\left(1-\frac{9}{n+5}\right)^{-9}}_{1}\right)^4 =$$

$$= \left(e^{-9}\right)^4 = e^{-36}$$

4, $$\lim_{n\to\infty}\left(\frac{n+2}{n+3}\right)^{3n} = \lim_{n\to\infty}\left(\frac{n+3-1}{n+3}\right)^{3n} = \lim_{n\to\infty}\left(\left(1-\frac{1}{n+3}\right)^n\right)^3 =$$

$$= \lim_{n\to\infty}\left(\left(1-\frac{1}{n+3}\right)^{n+3}\cdot\underbrace{\left(1-\frac{1}{n+3}\right)^{-3}}_{1}\right)^3 = e^{-3}$$

## Rekurzív sorozat határértéke

1, $a_1 = \sqrt{2}$ $\quad a_{n+1} = \sqrt{2+a_n}$ $\quad n \in \mathbb{N}$

$\lim_{n\to\infty} a_n = A$ $\quad \lim_{n\to\infty} a_{n+1} = A$

$$\sqrt{2+A} = A$$

$$2 + A = A^2$$

$$-A^2 + A + 2 = 0$$

$$\frac{-1 \pm \sqrt{1-4(-1)\cdot 2}}{-2} = \frac{-1\pm 3}{-2} = \begin{cases} -1 \\ 2 \end{cases}$$

$A_1 = -1$ $\cancel{4}$ mert $a_n < a_{n+1}$

$A_2 = 2$

2, $a_1 = 1 \quad a_{n+1} = \dfrac{2a_n}{n+1} \quad n \in \mathbb{N}$

$\lim\limits_{n\to\infty} a_n = A \quad \lim\limits_{n\to\infty} a_{n+1} = A$

$$\frac{2A}{n+2} = A$$

$$2A = An + 2A$$

$$0 = An$$

$$\underline{\underline{A = 0}}$$

$n = 0 \to$ ↯

Írja fel az alábbi sorozatok n-ik részletösszegét

1, $\sum\limits_{k=0}^{\infty} \dfrac{1}{2^k}$

Mj Mértani sor

$S_n = a_1 \cdot \dfrac{q^n - 1}{q - 1}$

$\to$ ha $\sum$ 0-tól kezd. akkor

$\cdot \dfrac{1}{1-q}$

$\to$ csak akkor konv. ha $|q| < 1$

$\sum\limits_{k=0}^{\infty} \dfrac{1}{2^k} = \sum\limits_{k=0}^{\infty} \left(\dfrac{1}{2}\right)^k \quad a_1 = 1$

$q = \dfrac{1}{2} \quad S_n = 1 \cdot \dfrac{1}{1-q} = 1 \cdot \dfrac{1}{1-\frac{1}{2}} = \underline{\underline{2}}$

2, $\sum\limits_{k=0}^{\infty} \left(-\dfrac{5}{4}\right)^k \quad q = -\dfrac{5}{4} \quad |q| > 1 \to$ a sor divergens

$S_n = \dfrac{1}{1+\frac{5}{4}} = \dfrac{4}{9}$

3, $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ -> parciális törtekre bontás

$$\frac{A}{k} + \frac{B}{k+1}$$

$$Ak + A + Bk = 1$$

$$Ak + Bk = 0k$$ -> k-nak itt ki kell esnie

$$A + B = 0$$

$$A = 1$$

$$\Rightarrow B = -1$$

$$\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{1}{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=2}^{\infty} \frac{1}{k} =$$

$$= 1 + \sum_{k=2}^{\infty} \frac{1}{k} - \sum_{k=2}^{\infty} \frac{1}{k} = \underline{\underline{1}}$$ ~~konvergens~~ konvergens!

4, $\sum_{k=1}^{n} \frac{3^{k+1}}{2^{2k}}$ $S_n = \frac{9}{4} \cdot \frac{\frac{3}{4}^n - 1}{\frac{3}{4} - 1}$ /

$\sum_{k=1}^{n} \frac{3^k \cdot 3^1}{(2^k)^2}$ $\frac{9}{4} + \frac{27}{16} + \frac{81}{64} - - -$

$$q = \frac{3}{4}$$

$$a_1 \cdot \frac{1}{1-q} = \frac{9}{4} \cdot \frac{1}{1-\frac{3}{4}} = \frac{9/4}{1/4} = \underline{\underline{9}}$$ $|q| < 1$

konvergens

Közönséges tört alak

0,78 123 123

$$= \frac{78}{100} + \frac{123}{100\,000} + \frac{123}{100\,000\,000} + \dots$$ $q = \frac{1}{1000}$

$$a_1 + \frac{1}{1-q}$$

$$\frac{78}{100}+\frac{\frac{123}{100\,000}}{1-\frac{1}{1000}}=\frac{78}{100}+\frac{\frac{123}{100\,000}}{\frac{999}{1000}}=\frac{78}{100}+\frac{\overset{41}{\cancel{123}}}{\cancel{100000}}\cdot\frac{\cancel{1000}}{\underset{333}{\cancel{999}}}=$$

$$=\frac{78}{100}+\frac{41}{33300}=\frac{5203}{6660}$$

Konvergens-e az alábbi végtelen sorok?

1, $\sum_{k=1}^{\infty}\frac{\cos^2(2k+1)}{(k^2+1)}<\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty$ Majoráns krit.

$\Rightarrow$ konvergens

2, $\frac{1}{3}+\frac{2^3}{3^2}+\frac{3^3}{3^3}+\frac{4^3}{3^4}+\dots=\sum_{k=1}^{\infty}\frac{k^3}{3^k}$

Hányados krit.:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=A$

$A<1$ konv

$A>1$ div

$A=1$ nem lehet eldönteni

$$\lim_{k\to\infty}\frac{\frac{(k+1)^3}{3^{(k+1)}}}{\frac{k^3}{3^k}}=\lim_{n\to\infty}\frac{k^3+3k^2+3k+1}{3\cdot\cancel{3^k}}\cdot\frac{\cancel{3^k}}{k^3}=$$

$$\lim_{k\to\infty}\frac{k^3+3k^2+3k+1}{3k^3}=\frac{1}{3}$$

$1>\frac{1}{3}\to$ konvergens

3, $\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{2}\right)}{k}$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\dots=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}$

$\to$ alternáló, [illegible]

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{2k-1}\right| = \lim_{k\to\infty}\frac{1^k\cdot 1^1}{2k-1} = \frac{1}{2k-1} = \underline{\underline{0}}$$

null sorozat

$\Rightarrow$ Leibniz sor $\Rightarrow$ Konvergens

4, $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{(-2)^n\cdot(n^2-n+1)}$

$\rightarrow$ alternáló

$$\frac{2}{-2}+\frac{5}{12}-\frac{10}{56}\dots \qquad |a_n| \text{ monoton fogyó}$$

$$\lim_{n\to\infty}\frac{n^2+1}{2^n\cdot(n^2-n+1)} = \lim_{n\to\infty}\underbrace{\frac{1}{2^n}}_{\downarrow\, 0}\cdot\underbrace{\frac{n^2+1}{n^2-n+1}}_{\downarrow\, 1} = \underline{\underline{0}}$$

null sorozat

Leibniz sor $\Rightarrow$ Konvergens!

5, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)}$ Majoráns krit.

(nem negatív) [illegible]

$$\sum_{n=1}^{\infty}\frac{1}{n(n+3)} < \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

$\Rightarrow$ Konvergens

6, $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2}^n}{(2n+1)!}$ $\qquad a_{n+1} = \dfrac{\sqrt{2}^{n+1}}{(2(n+1)+1)!} = \dfrac{\sqrt{2}^{n+1}}{(2n+3)!}$

$$\lim_{n\to\infty}\frac{\cancel{\sqrt{2}^n}\cdot\sqrt{2}}{(2n+3)!}\cdot\frac{(2n+1)!}{\cancel{\sqrt{2}^n}} = \lim_{n\to\infty}\frac{\sqrt{2}\,\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} = \underline{\underline{0}}$$

$A = 0 < 1 \Rightarrow$ Konvergens!

7, $\sum_{n=1}^{\infty} \frac{\binom{n}{2}}{\binom{n}{4}} = \sum_{n=1} \frac{\frac{n!}{2!(n-2)!}}{\frac{n!}{4!(n-4)!}} = 12 \sum_{n=1}^{\infty} \frac{\cancel{(n-4)!}}{(n-2)(n-3)\cancel{(n-4)!}} =$

$= 12 \sum_{n=1} \frac{1}{(n-2)(n-3)}$

$$\frac{1}{(n-2)(n-3)} = \frac{A}{(n-2)} + \frac{B}{(n-3)} = \frac{An-3A+Bn-2B}{(n-2)(n-3)}$$

$A + B = 0 \qquad A = -B$

$-3A \mp 2B = 1$

$3B - 2B = 1$

$B = 1 \qquad A = -1$

$$12 \sum_{n=1}^{\infty} \left( \frac{-1}{n-2} + \frac{1}{n-3} \right) = \cancel{12\left(-\frac{1}{2} - \frac{1}{1} + 1 + \frac{1}{2}\right.}$$

$$\cancel{12\left(\frac{1}{2} - 1 + \frac{1}{1} - \frac{1}{2}\right.}$$

## Határérték számítás (rendőr elv)

1, $\lim_{n \to \infty} \sqrt[n]{2^n + 5^n}$

$$\sqrt[n]{5^n} \leq \sqrt[n]{2^n + 5^n} \leq \sqrt[n]{5^n + 5^n}.$$

$$5 \leq \sqrt[n]{2^n + 5^n} \leq \sqrt[n]{2} + \sqrt[n]{5^n}$$

$\sqrt[n]{2} \to 1$, $\quad \sqrt[n]{5^n} \to 5$

$\lim \Rightarrow \underline{5}$

2, $\lim\limits_{n\to\infty} \sqrt[n]{n+1}$

$$\sqrt[n]{n} \le \sqrt[n]{n+1} \le \sqrt[n]{2n} = \sqrt[n]{2} \cdot \sqrt[n]{n}$$

$$\sqrt[n]{n} \to 1, \quad \sqrt[n]{2} \to 1, \quad \sqrt[n]{n} \to 1$$

$\Rightarrow \lim = \underline{\underline{1}}$

3, $\lim\limits_{n\to\infty} \sqrt[n]{n^6+2^n}$

$$\sqrt[n]{2^n} \le \sqrt[n]{n^6+2^n} \le \sqrt[n]{2\cdot 2^n} = \sqrt[n]{2} \cdot \sqrt[n]{2^n}$$

$$\sqrt[n]{2^n} \to 2, \quad \sqrt[n]{2} \to 1, \quad \sqrt[n]{2^n} \to 2$$

$\lim \Rightarrow \underline{\underline{2}}$

Adja meg a következő fv-ek határértékét!

1, $\lim\limits_{x\to 4} (6x+12) = \underline{\underline{36}}$

2, $\lim\limits_{x\to 3} \dfrac{2x^2-5x-3}{x^2+5} = \dfrac{18-18}{14} = \dfrac{0}{14} = \underline{\underline{0}}$

3, $\lim\limits_{x\to 2} \dfrac{x^3-3x^2+4}{x^3-2x^2+x-2} = \dfrac{8-12+4}{8-8+0} = \dfrac{0}{0}$

$\rightarrow \quad x^3-3x^2+4 : (x-2) = x^2-x-2$

$$\begin{array}{l} \underline{x^3-2x^2} \\ \quad -x^2+4 \\ \quad \underline{-x^2-2x} \\ \qquad 4-2x \\ \qquad \underline{-2x+4} \end{array}$$

$x^3-2x^2+x-2 : (x-2) = x^2+1$

$$\begin{array}{l} \underline{x^3-2x^2} \\ \qquad x-2 \\ \qquad \underline{x-2} \end{array}$$

$$\lim_{x\to 2} \frac{(x^2-x-2)\cancel{(x-2)}}{(x^2+1)\cancel{(x-2)}} = \frac{0}{5} = \underline{\underline{0}}$$

4, $\lim\limits_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} = \lim\limits_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} \cdot \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} \cdot \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4}$

$= \lim\limits_{x\to 0} \frac{x^2+1-1}{x^2+16-16} \cdot \frac{(\sqrt{x^2+16}+4)}{(\sqrt{x^2+1}+1)} = \frac{4+4}{1+1} = \underline{\underline{4}}$

5, $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim \frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x} = \lim\limits_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$

$= \lim\limits_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x} = ?$

$\downarrow$

$\frac{1}{2}$

6, $h(x) = \begin{cases} \frac{x^2-x-2}{x-2} & ha\ x \neq 2 \\ 1 & ha\ x = 2 \end{cases}$ $\quad x=2$ pontban

$\lim\limits_{x\to 2} \frac{x^2-x-2}{(x-2)} = \lim\limits_{x\to 2} \frac{(x+1)(x-2)}{(x-2)} = \underline{\underline{3}}$

$x^2-x-2 : x-2 = x+1$
$\underline{x^2-2x}$
$x-2$

7, $f(x) = \begin{cases} (1-x)^3 & ha\ x<0 \\ (1+x)^3 & ha\ x \geq 0 \end{cases}$

$\lim\limits_{x\to 0+} (1+x)^3 = 1$

$\lim\limits_{x\to 0} f(x) = \underline{\underline{1}}$

$\lim\limits_{x\to 0-} (1-x)^3 = 1$

8, $g(x)=\begin{cases} 1-x^2 & x\le 0 \\ (1-x)^2 & 0\le x\le 2 \\ 4-x & 2< x \end{cases}$   $0^-$   $0^+, 2^-$   $2^+$

$g(0)=1$   $g(2)=$

$\lim_{x\to 0^-} 1-x^2 = 1$   $\lim_{x\to 2^-} (1-x^2)^2 = 1$

$\lim_{x\to 0+} (1-x)^2 = 1$   $\lim_{x\to 2^+} (4-x) = 2$

$\to$ balról folytonos, 1-nél szakadása van

**Fv határérték (L'Hopital szabállyal)**

L'Hopital

$$\lim_{x\to n} \frac{f(x)}{g(x)} = \lim_{x\to n} \frac{f'(x)}{g'(x)}$$

1, $\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{0}{0}$

$$\Rightarrow \lim_{x\to 0} \frac{\sin'(ax)}{\sin'(bx)} = \lim_{x\to 0} \frac{\overset{1}{\cos(ax)}\cdot a}{\underbrace{\cos(bx)}_{1}\cdot b} = \frac{a}{b}$$

2, $\lim_{x\to 0} \frac{1-\cos(x)}{x} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{(1-\cos(x))'}{x'} = \sin x = 0$$

3, $\lim_{x\to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{(\ln x)'}{x'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

4, $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{\sin x - x} = \dfrac{0}{0}$

$$\lim_{x \to 0} \frac{(\cos x - 1)'}{(\sin x - x)'} \underset{LH}{=} \lim \frac{(-\sin x)'}{(\cos x - 1)'} \underset{LH}{=} \lim_{x \to 0} \frac{-\cos x}{-\sin x} =$$

$$= \lim_{x \to 0} ctg(x) = \underline{\underline{0}}$$

**Határozzuk meg a következő fv inverzét**

$$f(x) = \frac{x+1}{x-1}$$

$$x = \frac{y+1}{y-1}$$

$$x = \frac{y-1+2}{y-1} = 1 + \frac{2}{y-1}$$

$$xy - x = y - 1 + 2$$

$$xy - y = x + 1$$

a fv tengelyesen szimm az xy tengelyre

$$y(x-1) = x + 1$$

$$y = \frac{x+1}{x-1}$$

**Deriválja az alábbi fv-eket**

1, $f(x) = (x^3+1)^2$

$$f'(x) = 2(x^3+1) \cdot 3x^2$$

2, $f(x) = \sin\sqrt{x+3}$

$$f'(x) = \cos\sqrt{x+3} \cdot \frac{1}{2}(x+3)^{-\frac{1}{2}} \cdot 1$$

Végezzen teljes fv vizsgálatot!

1, $f(x) = x^3 - 2x$

$x^3 - 2x = 0$ -> itt metszi a fv az x tengelyt

$x(x^2 - 2) = 0$

$\boxed{x_1 = 0}$

$x^2 - 2 = 0$

$x^2 = 2$

$\boxed{x_2 = \sqrt{2}}$ $\boxed{x_3 = -\sqrt{2}}$

$f'(x) = 3x^2 - 2$

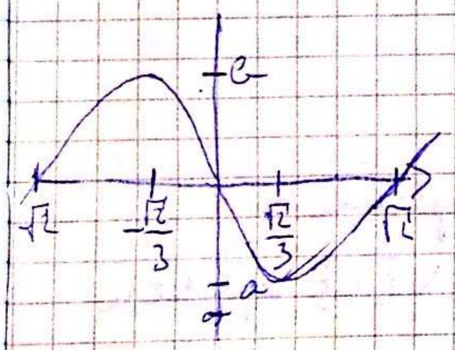
$3x^2 - 2 = 0$ -> itt van a fv-nek lokális szélsőértéke

$x^2 = \frac{2}{3}$

$x_{1,2} = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{2}}{3}$

$f\left(\frac{\sqrt{2}}{3}\right) = \left(\frac{\sqrt{2}}{3}\right)^3 - 2\frac{\sqrt{2}}{3}$ => lok min ~a

$f\left(-\frac{\sqrt{2}}{3}\right) =$ lok max ~b




| $x < -\frac{\sqrt{2}}{3}$ | $x = -\frac{\sqrt{2}}{3}$ | $-\frac{\sqrt{2}}{3} < x < \frac{\sqrt{2}}{3}$ | $x = \frac{\sqrt{2}}{3}$ | $x > \frac{\sqrt{2}}{3}$ |
|---|---|---|---|---|
| ↗ | lok max | ↘ | lok min | ↗ |

$f''(x) = 6x$

| | $x < 0$ | $x = 0$ | $x > 0$ |
|---|---|---|---|
| $f''$ | − | 0 | + |
| | konkáv | infl. pont | konvex |

$R_f \in \mathbb{R}$

$D_f \in \mathbb{R}$

2, $f(x) = x \cdot e^{-x}$

$$x \cdot e^{-x} = 0$$

$x = 0 \to$ itt metszi az x-tengelyt

$f'x = e^{-x} - x \cdot e^{-x}$

$$e^{-x} - e^{-x} \cdot x = 0$$

$$e^{-x} = e^{-x} \cdot x$$

$1 = x \to$ lok. szé.

$f(1) = 1 \cdot e^{-1} = \frac{1}{e}$

| | $x < 1$ | $x = 1$ | $x > 1$ |
|---|---|---|---|
| $f'$ | + | 0 | − |
| $f$ | ↗ | $lok_{max}$ | ↘ |

$f''(x) = -e^{-x} - e^{-x} + x e^{-x} = 0$

$$-1 - 1 + x = 0$$

$x = 2 \to$ inflexiós pont

Végezzünk folytonosság vizsgálatot az alábbi fv-eken.

1, $f(x) = \begin{cases} \frac{3}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$

$\lim\limits_{x \to 1+} \frac{3}{x-1} = \frac{3}{0} = \infty$ A fv folytonos $\forall x \in \mathbb{R} \setminus \{1\}$ esetén

$\lim\limits_{x \to 1-} \frac{3}{x-1} = \frac{3}{0} = -\infty$ másodfajú szakadás $x_0 = 1$ pontban

2, $f(x) = \begin{cases} \frac{1}{2} x^2 & x \leq 2 \\ x & x > 2 \end{cases}$

$\lim\limits_{x \to 2-} \frac{1}{2} x^2 = 2 = \lim\limits_{x \to 2+}$ A fv folytonos $\forall$

$\Rightarrow \exists \lim\limits_{x \to 2}$ ... $x \in \mathbb{R}$-en

3, $f(x) = \begin{cases} 1-x^2 & x \le 0 \\ (1-x)^2 & 0 < x \le 2 \\ 4-x & x > 2 \end{cases}$

$\lim\limits_{x \to 0-} 1-x^2 = 1$

$\lim\limits_{x \to 0+} (1-x)^2 = 1$ $\qquad \lim\limits_{x \to 2-} (1-x)^2 = 1$

$\lim\limits_{x \to 2+} 4-x = 2$

jobb és bal oldali határértékek $\exists$ -nek $x_0 = 2$ pontban, de nem egyeznek meg

$\downarrow$

első fajú nem megszüntethető szakadás

A fv- folytonos $\forall\, x \in \mathbb{R} \setminus \{2\}$ pontban

## Taylor Formula

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

1, $x_0 = 0$

$f(x) = e^x \qquad T_1(x) = ? \qquad f'x = e^x$

$$T_1(0) = \frac{f'(0)}{1}(x-0)^1 = e^0 \cdot x = \underline{\underline{x}}$$

2, $f(x) = \sin(x)$ ; $x_0 = 0$ ; $T_5(x) = ?$

| $k$ | $f^{(k)}$ | $f^{(k)}(x_0)$ |
|---|---|---|
| 0 | $\sin x$ | 0 |
| 1 | $\cos x$ | 1 |
| 2 | $-\sin x$ | 0 |
| 3 | $-\cos x$ | $-1$ |
| 4 | $\sin x$ | 0 |
| 5 | $\cos x$ | 1 |

$$T_5(x) = 0 + x + 0 - \frac{1}{6}x^3 + 0 + \frac{1}{120}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

3, $f(x) = 1 + \frac{3}{x}$ , $x_0 = 1$ , $T_4(x) = ?$  $x \in \mathbb{R} \setminus \{0\}$

| $k$ | $f^{(k)}$ | $f^{(k)}(x_0)$ |
|---|---|---|
| 0 | $1 + 3x^{-1}$ | 4 |
| 1 | $-3x^{-2}$ | $-3$ |
| 2 | $6x^{-3}$ | 6 |
| 3 | $-18x^{-4}$ | $-18$ |
| 4 | $72x^{-5}$ | 72 |

$$T_4(x) = 4 - 3(x-1) + 3(x-1)^2 - 3(x-1)^3 + 3(x-1)^4$$

4, $\sin(62^\circ)$ értéke, minél jobb közelítéssel
hiba $< 10^{-6}$

$x_0 = \frac{\pi}{3}$

$T_n\left(\frac{\pi}{3}\right) = ?$

| $k$ | $f^{(k)}$ | $f^{(k)}(x_0)$ |
|---|---|---|
| 0 | $\sin x$ | $\frac{\sqrt{3}}{2}$ |
| 1 | $\cos x$ | $0{,}5$ |
| 2 | $-\sin x$ | $-\frac{\sqrt{3}}{2}$ |
| 3 | $-\cos x$ | $-0{,}5$ |
| 4 | $\sin x$ | $\frac{\sqrt{3}}{2}$ |

$$T_3\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \cancel{0,5}\,\frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x - \frac{\pi}{3}\right)^3$$

$\Rightarrow$ $x = 62^\circ$ esetén kapjuk meg az értéket

$$\xi \in [x, x_0] = [62^\circ, 60^\circ]$$

$$L_3 = \frac{\sqrt{3}}{48}\left(\cancel{x}\,\xi - \frac{\pi}{3}\right)^4 = \frac{\sqrt{3}}{48}(61-60)^4 = \frac{\sqrt{3}}{48}\left(\frac{\pi}{180}\right)^4 = \underline{4{,}9 \cdot 10^{-15}}$$ ✓

hiba

Határozza meg az alábbi fv-ek primitív fv-ét

1, $f(x) = \frac{x+1}{x-1}$  $F(x) = ?$

$$\int \frac{x+1}{x-1}\,dx = \int \frac{x-1+2}{x-1}\,dx = \int 1\,dx + \int \frac{2}{x-1}\,dx =$$

$$= x + 2\ln(x-1) + C$$

2, $f(x) = \frac{\ln^5 x}{x}$ [illegible]

$$\int \frac{\ln^5(x)}{x}\,dx = \int \underset{g'}{\frac{1}{x}} \cdot \underset{g}{\ln^5(x)}\,dx = \frac{\ln^6 x}{6} + C$$

3, $f(x) = \frac{e^{3x}}{e^{3x}+5}$

$$\int \frac{e^{3x}}{e^{3x}+5} = \int \underset{g}{e^{3x}} \cdot \underset{g^{-1}}{\frac{1}{e^{3x}+5}}\,dx = \frac{\ln(e^{3x}+5)}{3}$$

4,

$$\int x \sin(x^2)\,dx = \frac{-\cos x^2}{2}$$

**Számítsa ki az alábbi fv-ek határozott integrálját a megadott intervallumon**

1, $\int_0^4 5x^3 + 1\,dx = \left[5\frac{x^4}{4} + x\right]_0^4 = \left(5\frac{4^4}{4} + 4\right) = 324$

2, $\int_0^{\frac{\pi}{2}} \sin(8x)\,dx = \left[-\frac{\cos(8x)}{8}\right]_0^{\frac{\pi}{2}} = -\frac{1}{8} - \left(-\frac{1}{8}\right) = 0$

3, $\int_0^1 \frac{1}{3x+5}\,dx = \left[\frac{1}{3}\ln(3x+5)\right]_0^1 = \frac{1}{3}\ln(8) - \frac{1}{3}\ln(5)$

4, $\int_5^7 \frac{x^2-5x+9}{x}\,dx = \int_5^7 \underset{f'}{\frac{1}{x}} \cdot \underset{g}{(x^2-5x+9)} = \cancel{\ldots}$

$f = \ln x \qquad g' = \cancel{\frac{x^3}{3} - \frac{5x^2}{2}} + 9x$

$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 9\right]_5^7 - \int_5^7 \ln x \left(\frac{x^3}{3} - \frac{5x^2}{2} + 9x\right)$

## Parciálisan integrálás

$fg - \int f$

1. $\int_1^e x \cdot \ln(x)\,dx = \left[\ln x \cdot \frac{x^2}{2}\right]_1^e - \underbrace{\int_1^e \frac{x^2}{2x}\,dx}_{\frac{1}{2}\int_1^e x} =$

$g'$ $f$

$g = \frac{x^2}{2}$ $f' = \frac{1}{x}$

$= \left(\underbrace{\ln e}_{1} \cdot \frac{e^2}{2}\right) - \left(\underbrace{\ln 1}_{0} \cdot \frac{1}{2}\right) - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e =$

$= \frac{e^2}{2} - \frac{1}{2}\frac{e^2}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} + C$

2. $\int_1^2 x^2 e^x\,dx = \left[x^2 e^x\right]_1^2 - \int_1^2 2x\, e^x\,dx =$

$g$ $f'$

$g' = 2x$ $f = e^x$

$f$ $g'$

$f' = 2$ $g = e^x$

$\left[2x e^x\right]_1^2 - \int_1^2 2e^x\,dx$

$2e^x \Big|_1^2$

$= 4e^2 - e - \left(4e^2 - 2e - \left(2e^2 - 2e\right)\right) =$

$= \cancel{4e^2} - e - \cancel{4e^2} + \cancel{2e} + 2e^2 - \cancel{2e} = \underline{\underline{2e^2 - e}}$

3. $\int_0^{\frac{\pi}{4}} x \cdot \sin(x)\,dx = \left[x(-\cos x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\cos(x)\,dx =$

$f$ $g'$

$f' = 1$ $g = -\cos(x)$

$$= \frac{\pi}{4} - \frac{\sqrt{2}}{2} + \left[\sin(x)\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \underline{\underline{\frac{\pi}{4}}}$$

## Racionális tört fv integrálja

$$\int \frac{1}{(x-2)(x-4)}\,dx = \int \frac{A}{x-2} + \frac{B}{x-4}\,dx = \int -\frac{1}{2}\cdot\frac{1}{(x-2)}\,dx + \int \frac{1}{2}\,\frac{1}{x-4}\,dx =$$

$$Ax - 4A + Bx - 2B = 1$$

$$A + B = 0$$

$$A = -B$$

$$-4A - 2B = 1$$

$$4B - 2B = 1$$

$$B = \frac{1}{2} \qquad A = -\frac{1}{2}$$

$$= -\frac{1}{2}\ln(x-2) + \frac{1}{2}\ln(x-4) + C$$

## Érintő egyenes egyenlete

$$y = m(x - x_0) + y_0$$

1, $f(x) = \sin x^2$

$$x_0 = \frac{\sqrt{\pi}}{2}$$

$$y_0 = f(x_0) = \sin\left(\frac{\sqrt{\pi}}{2}\right)^2 = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$f'x = \cos x^2 \cdot 2x$$

$$m = f'(x_0) = \cos\frac{\pi}{4}\cdot\frac{2\sqrt{\pi}}{2} = \frac{\sqrt{2}}{2}\cdot\sqrt{\pi}$$

$$y = \frac{\sqrt{2}}{2}\sqrt{\pi}\left(x - \frac{\sqrt{\pi}}{2}\right) + \frac{\sqrt{2}}{2}$$

2, $y = m(x - x_0) + y_0$

$f(x) = 2x^2 + 3x^3$

$y = x \Rightarrow m = 1$

$f'x = 4x + 9x^2$

$4x + 9x^2 = 1$

$9x^2 + 4x - 1 = 0$

$x_{1,2} = \frac{-4 \pm \sqrt{16 - 4(-9)(-1)}}{2(-9)} = \frac{-4 \pm \sqrt{16-36}}{-18}$ ↯

3, $f(x) = 3x - x^2$

$x_0 = 1$

$y_0 = f(x_0) = 3 - 1 = 2$ $\quad f'x = 3 - 2x$

$m = f'(x_0) = 3 - 2 = 1$

$y = (x - 1) + 2 = x + 1$

## Implicit fv deriválása

1, $f(x) = y^4 + 3y - 4x^3 = 5x + 1$ $\quad f'(x) = ?$ $\quad y = f(x)$

$f^4(x) + 3f(x) - 4x^3 = 5x + 1$

$4f^3(x) \cdot f'(x) + 3 \cdot f'(x) - 12x^2 = 5$

$f'(x)\left(4f^3(x) + 3\right) = 5 + 12x^2$

$f'(x) = \frac{5 + 12x^2}{4f^3(x) + 3} = \frac{5 + 12x^2}{4y^3 + 3}$

2/ $x^2+y^2=1$

$x^2+f^2(x)=1$

$2x+2f(x)\cdot f'(x)=0$

$f'(x)=\frac{-2x}{2f(x)}=-\frac{x}{y}$

3/ $x^3+y^3=3xy$

$x^3+f^3(x)=3x\cdot f(x)$

$3x^2+3f^2(x)\cdot f'(x)=3f(x)+3xf'(x)$

$-3xf'(x)+3f^2(x)\cdot f'(x)=3f(x)-3x^2$

$f'(x)=\frac{3f(x)-3x^2}{3f^2(x)-3x}=\frac{3y-3x^2}{3y^2-3x}$

Írja fel az alábbi diff-egyenlet megoldását

1/ $y'=-2xy \qquad y(0)=2$

$\frac{dy}{dx}=-2xy$

$\int\frac{1}{y}dy=\int-2x\,dx$

$\ln y=-x^2 \qquad y=e^{-x^2+C}=e^{-x^2}\cdot e^{C}$

$2=e^{0+c}=e^{0+\ln 2}$

$c=\ln 2 \Rightarrow y=e^{-x^2+\ln 2}=2e^{-x^2}$

2/ $y'=\frac{y}{x} \qquad y(1)=1$

$\frac{dy}{dx}=\frac{y}{x}$

$\int\frac{1}{y}dy=\int\frac{1}{x}dx \qquad \ln y=\ln x+C$

$$e^{\ln y} = e^{\ln x + c} = e^{\ln x} \cdot e^{c}$$

$$y = x \cdot e^{c}$$

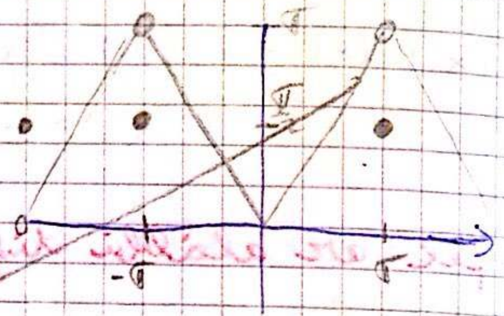
$$1 = 1 \cdot e^{c} \qquad c = \ln 1$$

$$y = x \cdot e^{\ln 1} \Rightarrow y = x$$

Írja fel az alábbi fv-ek Fourier ~~transzformáltját~~ sorát

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

1, $f(x) = \begin{cases} -x & \text{ha } -\pi < x \le 0 \\ x & 0 < x < \pi \\ \frac{\pi}{2} & x = \pi \end{cases}$



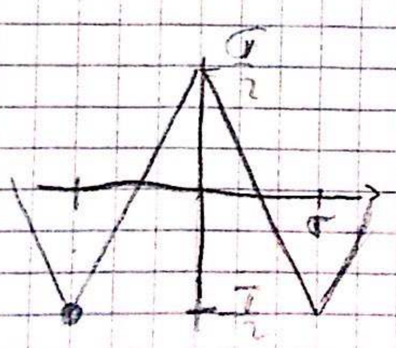
páros fv $\quad b_k = 0$

$$a_k = \frac{1}{\pi} \int_0^{\pi} x \cos kx \, dx = \frac{1}{\pi} \left( \left[ x \frac{\sin kx}{k} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin kx}{k} dx \right) =$$

$f = x \quad g' = \cos kx$

$f' = 1 \quad g = \frac{\sin kx}{k}$

1, $f(x) = \begin{cases} x + \frac{\pi}{2} & -\pi \le x \le 0 \\ \frac{\pi}{2} - x & 0 \le x \le \pi \end{cases} \qquad f(x + 2k\pi)$



páros fv

$b_k = 0$

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{0} x + \frac{\pi}{2}\,dx + \int_0^{\pi} \frac{\pi}{2} - x\,dx\right) = \frac{1}{\pi}\left(\left[\frac{x^2}{2} + \frac{\pi}{2}x\right]_{-\pi}^{0} + \left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^{\pi}\right) =$$

$$\frac{1}{\pi}\left(-\underbrace{\frac{\pi^2}{2}}_{(-\pi)^2=\pi^2} - \frac{\pi^2}{2} + \frac{\pi^2}{2} + \frac{\pi^2}{2}\right) = \underline{0}$$

$[fg] - \int f'g$

$$a_k = \frac{1}{\pi}\left(\int_{-\pi}^{0}\underbrace{\left(x+\frac{\pi}{2}\right)}_{f}\underbrace{\cos kx}_{g'}\,dx + \int_0^{\pi}\underbrace{\left(\frac{\pi}{2}-x\right)}_{f}\underbrace{\cos kx}_{g'}\,dx\right) =$$

$$= \cancel{\frac{1}{\pi}\left(\left[\left(x+\frac{\pi}{2}\right)\frac{\sin kx}{k}\right]_{-\pi}^{0} + \left(\frac{\pi}{2}-x\right)\frac{\sin kx}{k} - \int_{-\pi}^{0}\frac{\sin kx}{k} + \left(\left[\frac{\pi}{2}-x\right]\frac{\sin kx}{k}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin kx}{k}\right)\right) = \frac{1}{\pi}\left(-\left(\pi + \frac{\pi}{2}\right.\right.}$$

$$a_k = \frac{1}{\pi}\left(\int_{-\pi}^{0}\underbrace{x}_{f}\,\underbrace{\cos kx}_{g'}\,dx + \int_{-\pi}^{0}\frac{\pi}{2}\cos kx\,dx + \int_0^{\pi}\frac{\pi}{2}\cos kx\,dx - \int_0^{\pi} x\cos kx\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[x\frac{\sin kx}{k}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin kx}{k} + \left[\frac{\pi}{2}\frac{\sin kx}{k}\right]_{-\pi}^{0} + \left[\frac{\pi}{2}\frac{\sin kx}{k}\right]_0^{\pi} - \left[x\frac{\sin kx}{k}\right]_0^{\pi} +\right.$$

$$\left.+\int_0^{\pi}\frac{\sin kx}{k}\right) = \frac{1}{\pi}\left(\cancel{\pi\frac{\sin k\pi}{k}} - \left[-\frac{\cos kx}{k^2}\right]_{-\pi}^{0} - \cancel{\frac{\pi}{2}\frac{\sin k(-\pi)}{k}} + \cancel{\frac{\pi}{2}\frac{\sin k(\pi)}{k}} -\right.$$

$$\left.-\cancel{\pi\frac{\sin k\pi}{k}} + \left[-\frac{\cos kx}{k^2}\right]_0^{\pi}\right) = \frac{1}{\pi}\left[\frac{1}{k^2} - \frac{\cos k(-\pi)}{k^2} - \frac{\cos k(\pi)}{k^2} + \frac{1}{k^2}\right] =$$

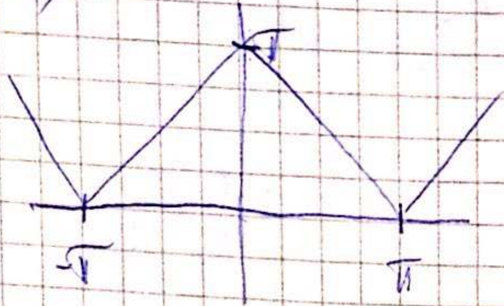
$$= \frac{2}{\pi k^2}\left(1 - \cos k(\pi)\right) = \begin{cases} 0 & k = 2n \\ \dfrac{4}{\pi k^2} & k = 2n+1 \end{cases}$$

$\cos k(\pi) = 1$

$\cos k(\pi) = -1$

$$f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$$

2/



$$f(x) = \begin{cases} \pi - x & 0 \le x \le \pi \\ x + \pi & -\pi \le x \le 0 \end{cases}$$

páros fv. $b_k = 0$

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{0} (x+\pi)\,dx + \int_0^{\pi} \pi - x\,dx\right) = \frac{1}{\pi}\left(\left[\frac{x^2}{2} + \pi x\right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} - \pi^2 + \pi^2 - \frac{\pi^2}{2}\right) = \cancel{\frac{\pi^2}{\pi}} \; 0$$

$$a_k = \frac{1}{\pi}\left(\int_{-\pi}^{0} x \cos kx\,dx + \int_{-\pi}^{0} \pi \cos kx\,dx + \int_0^{\pi} \pi \cos kx\,dx - \int x \cos kx\,dx\right.$$

$$= \frac{1}{\pi}\left(\left[x\frac{\sin kx}{k}\right]_{-\pi}^{0} - \int_{-\pi}^{0} \frac{\sin kx}{k}\,dx + \left[\pi \frac{\sin kx}{k}\right]_{-\pi}^{0} + \left[\pi \frac{\sin kx}{k}\right]_0^{\pi} -\right.$$

$$\left. - \left[x\frac{\sin kx}{k}\right]_0^{\pi} + \int_0^{\pi} \frac{\sin kx}{k}\,dx\right) = \frac{1}{\pi}\left(\cancel{\pi \frac{\sin k(-\pi)}{k}} - \left[\frac{-\cos kx}{k^2}\right]_{-\pi}^{0} +\right.$$

$$\left. \cancel{\pi \frac{\sin k(-\pi)}{k}} + \cancel{\pi \frac{\sin k(\pi)}{k}} - \cancel{\frac{\pi \sin k(\pi)}{k}} + \left[\frac{-\cos kx}{k^2}\right]_0^{\pi}\right) =$$

$$= \frac{1}{\pi}\left(+\left[\frac{\cos kx}{k^2}\right]_{-\pi}^{0} - \left[\frac{\cos kx}{k^2}\right]_0^{\pi}\right) =$$

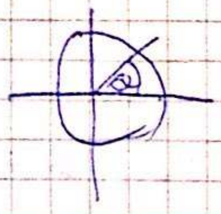
$$= \frac{1}{\pi}\left(\frac{1}{k^2} - \frac{(-1)^k}{k^2} - \frac{(-1)^k}{k^2} + \frac{1}{k^2}\right) = \frac{2}{\pi k^2}\left(1 - (-1)^k\right) =$$

$$= \begin{cases} 0 & k = 2n \\ \dfrac{4}{k^2 \pi} & k = 2n+1 \end{cases}$$

$$f(x) = \frac{4}{\pi} \sum^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}$$

Adja meg paraméteresen az alábbi $\mathbb{R}^2$-beli görbéket

1, origó középpontú $r > 0$ sugarú kör

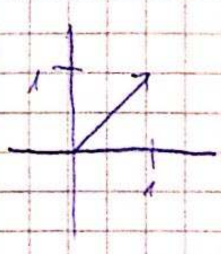


$t = \theta \qquad t \in [0, 2\pi]$

$x = r\cos\theta \qquad x(t) = r\cos t$

$y = r\sin\theta \qquad y(t) = r\sin t$

2, origót és az $(1,1)$ pontot összekötő szakaszt



$x(t) = t \qquad 0 \leq t \leq 1$

$y(t) = t$

Adja meg polár-koordinátákkal a következő pontokat

1, $P_1(2, 2\sqrt{3})$

$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$

$y = r\sin\theta \qquad \theta = \operatorname{arctg}\left(\frac{y}{x}\right)$

$r = \sqrt{4 + 12} = 4$

$\theta = \operatorname{arctg}\left(\frac{2\sqrt{3}}{2}\right) = 60° = \frac{\pi}{3}$

$x = 4\cos(60°) = 4\cos\frac{\pi}{3}$

$y = 4\sin(60°) = 4\sin\frac{\pi}{3}$

Adja meg Desc. illetve polárkoord.-kal az alábbi síkbeli tartományokat

1, origó kp-ú $a > 0$ sugarú zárt körlemez

polár $\qquad R = 2a$

$\theta \in [0, 2\pi] \qquad 0 \leq r \leq R$

Descartes

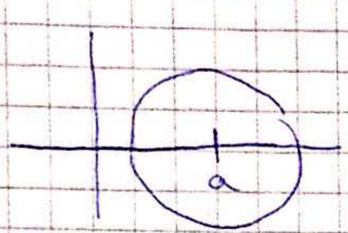
$-R \leq x \leq R$

$-\sqrt{R^2-x^2} \leq y \leq \sqrt{R^2-x^2}$

2, $(a,0)$ kp-ú, $a>0$ sugarú, zárt körlemez

Des:

$0 \leq x \leq 2a \qquad R=a$



$-\sqrt{a^2-(x-a)^2} \leq y \leq \sqrt{a^2-(x-a)^2}$

Polar:

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$0 \leq r \leq 2a\cos\theta$

3, y tengellyel párhuzamos, tőle $a>0$ távolságra lévő egyenes és az y tengely közé eső zárt tart.

Des:

$0 \leq x \leq a$

$-\infty \leq y \leq \infty$

Pol:

$0 \leq r \leq \frac{a}{\cos}$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Legyen $A \in \mathbb{R}$ rögzített, ábrázoljuk a következő fv-t

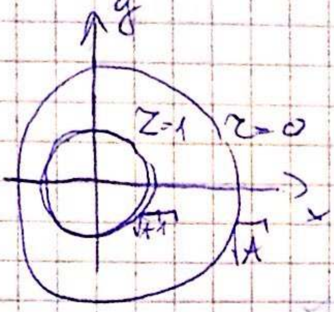
$f(x,y) = A - x^2 - y^2 \qquad (x,y) \in \mathbb{R}^2$

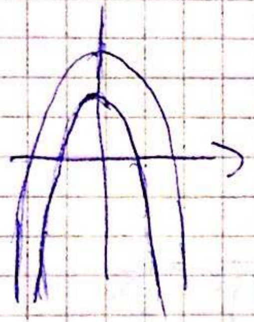
$z = A - x^2 - y^2$

$z = 0 \qquad A - x^2 - y^2 = 0 \qquad A = x^2 + y^2 \rightarrow r = \sqrt{A}$

$z = 1 \qquad A - x^2 - y^2 = 1 \qquad A - 1 = x^2 + y^2 \rightarrow r = \sqrt{A-1}$

$\sqrt{A-1}^2 = x^2 + y^2$

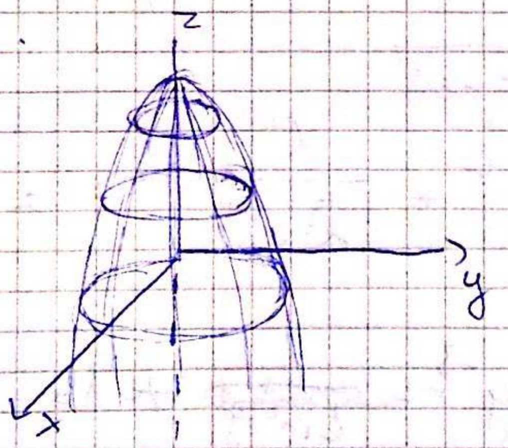


$x = 0 \qquad z = A - y^2$

$x = 1 \qquad z = A - 1 - y^2$

grafikon:



# Parciális deriváltak

$tg' = \frac{1}{\cos^2 x}$

1, $f(x,y) = tg(3x-5y)$

$f'x = \frac{1}{\cos^2(3x-5y)} \cdot 3$

$f'y = \frac{1}{\cos^2(3x-5y)} \cdot (-5)$

2, $f(x,y) = \ln\sqrt{x^7 y^4}$

$\cancel{f'x = \frac{1}{\sqrt{x^7 y^4}} \cdot \frac{1}{2} x^{7(-\frac{1}{2})} \cdot y^{4} \cdot 7}$

3, $f(x,y) = arctg\left(\frac{y}{x}\right)$

$f'x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot -\frac{y}{x^2}$ $\quad f'y = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$

4, $f(x,y) = \ln\frac{x^7}{y^2} + \sqrt{x^2-e^y}$

$f'x = \frac{1}{\frac{x^7}{y^2}} \cdot \frac{1}{y^2} \cdot 7x^6 + \frac{1}{2\sqrt{x^2-e^y}} \cdot 2x = 7\frac{1}{x} + (x^2-e^y)^{-\frac{1}{2}} x$

$f'y = \frac{1}{\frac{x^7}{y^2}} \cdot x^7 \cdot (-2) y^{-3} + \frac{1}{2\sqrt{x^2-e^y}} (-)e^y = -2\frac{1}{y} + \frac{1}{2}(x^2-e^y)^{-\frac{1}{2}}(-e^y)$

$f'x'x = -7x^{-2} + \left(-\frac{1}{2}\right)(x^2-e^y)^{-\frac{3}{2}} \cdot 2x^2 + (x^2-e^y)^{-\frac{1}{2}} \cdot 1$

$f'x'y = -\frac{1}{2}(x^2-e^y)^{-\frac{3}{2}} \cdot (-e^y) \cdot x$

$f'y'y = 2y^{-2} + -\frac{1}{4}(x^2-e^y)^{-\frac{3}{2}}(-e^y) \cdot (-e^y) + \frac{1}{2}(x^2-e^y)^{-\frac{1}{2}}(-e^y)$

$f'y'x = (-e^y) - \frac{1}{4}(x^2-e^y)^{-\frac{3}{2}} \cdot 2x$

Határozzuk meg az alábbi fv iránymenti deri váltják

1, $f(x,y) = x^3y^2 \qquad P_0(-1,2)$

$$V_0 = \frac{(4,-3)}{\sqrt{16+9}} = \qquad v=(4,-3) \rightarrow \text{irány}$$

$$= \left(\frac{4}{5}, -\frac{3}{5}\right)$$

$$\nabla f(x,y) = (f'x, f'y) = (3x^2y^2, 2x^3y)$$

$P_0$ helyen $\rightarrow (3(-1)^2 4, 2(-1)^3 2) = (12,-4)$

$$Dvf(-1,2) = \langle(-12,4), \left(\frac{4}{5}, -\frac{3}{5}\right)\rangle = \frac{48}{5} + \frac{12}{5} = \underline{\underline{12}}$$

$$\langle \nabla f, V_0 \rangle$$

2, $\alpha = 120°$

$$v = (\cos 120°, \sin 120°) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \rightarrow V_0$$

$$Dvf = \langle(-12,4), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\rangle = -6 - 2\sqrt{3}$$

3, Az $A(1,2)$ pontból a $B(2,5)$ pontba mutató irány

$$\underline{V} = (1, 3)$$

$$V_0 = \frac{(1,3)}{\sqrt{1+9}} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

$$D\nabla f(-1,2) = \langle(-12,4), \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)\rangle = -\frac{12}{\sqrt{10}} + \frac{12}{\sqrt{10}} = \underline{\underline{0}}$$

4, $f(x,y) = x^2 + 3y^2 \quad P_0 = (1,2) \quad \alpha = \frac{\pi}{3}$

$f'_x = 2x \qquad \operatorname{grad} f(P_0) = (2, 12)$

$f'_y = 6y \qquad v = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$D_v f = \left\langle (2,12) ; \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \right\rangle = 1 + 6\sqrt{3}$

## Érintősík egyenlete

1, $f(x,y) = x^2 + 3y^2 \quad P_0(1,2)$

$f'_x = 2x$

$f'_y = 6y \qquad f(P_0) = 13$

Egyenlet

$\nabla f(P_0) = (\underbrace{2}_{f'_x(P_0)}, \underbrace{12}_{f'_y(P_0)}, -1) \qquad 2x + 12y - z = 2 + 24 - 13 = 13$

$2x + 12y - z = 13$

2, $f(x,y) = x^2 + y^2 \quad P_0(1,1,2)$

$f'_x = 2x$

$f'_y = 2y$

$\nabla f(P_0) = (2, 2, -1) \qquad 2x + 2y - z = 0$

Határozzuk meg az alábbi fv-ek lokális szé-t

1, $f(x,y) = x^2 - 4xy + y^3 + 4y$

$f'_x = 2x - 4y$

$f'_y = -4x + 3y^2 + 4$

$2x - 4y = 0$

$2x = 4y$

$x = 2y$

$-4(2y) + 3y^2 + 4 = 0$

$-8y + 3y^2 + 4 = 0$

$y_{1,2} = \frac{8 \pm \sqrt{64 - 4 \cdot 3 \cdot 4}}{2 \cdot 3} = \frac{8 \pm 4}{6} = \begin{cases} 2 = y_1 \\ \frac{2}{3} = y_2 \end{cases}$

$x_1 = 4$

$x_2 = \frac{4}{3}$

$f''_{xx} = 2$ $\quad f''_{yy} = 6y$

$f''_{xy} = -4$ $\quad f''_{yx} = -4$

$H(P_1) = \begin{bmatrix} 2 & -4 \\ -4 & 12 \end{bmatrix}$

$\det H = 24 - 16 = 8 \quad > 0$

$\exists$ szé

pozitív definit

lok min.

$H(P_2) = \begin{bmatrix} 2 & -4 \\ -4 & 4 \end{bmatrix}$

$\det H_2 = 8 - 16 = -8$

$\to P_2$ nyeregpont

2/ $f(x,y) = x^3 + 3xy + y^3$

$f'_x = 3x^2 + 3y$

$f'_y = 3x + 3y^2$

$3x^2 + 3y = 0$

$3x^2 = -3y$

$x^2 = -y$

$x = -y^2 \qquad y = -x^2$

$x = (-x^2)^2$

$x + x^4 = 0$

$x(1 + x^3) = 0$

$x_1 = 0 \qquad y_1 = 0 \qquad P_1(0,0)$

$x_2 = -1 \qquad y_2 = -1 \qquad P_2(-1,-1)$

$\cancel{H_1 = [}$

$f'_x{}'_x = 6x \qquad f'_x{}'_y = 3$

$f'_y{}'_y = 6y \qquad f'_y{}'_x = 3$

$H_{1_{P_1}} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ $\det H_1 = -9$

nyeregpont

$H_2 = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$ $\det H_2 = 27 > 0 \rightarrow \exists$ sz.e.

negatív def.

$P_2$ lokális maximum

Deriválja az alábbi implicit fv-t!

1, $xy^2 + 3x^2y - 5x = 0$  $y = f(x)$

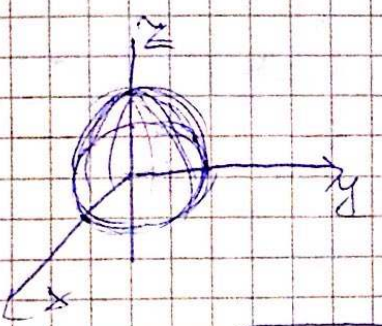
$x \cdot f^2(x) + 3x^2 \cdot f^2(x) - 5x = 0$

$f^2(x) + x \cdot 2f'(x) + 6x\, f^2(x) + 3x^2 \cdot 2f'(x) - 5 = 0$

$f^2(x)\left(1 + 6x\right) + f'(x)\left(2x + 6x^2\right) \cdot 5 = 0$

$$f'(x) = -\frac{f^2(x) + 6x f^2(x)}{2x + 6x^2} = -\frac{y^2 + 6xy^2 + 5}{2x + 6x^2}$$

Írja fel a gömbi polárkr.-k segítségével az origó kp-ú egységgömb felső térrészébe eső felét.



$x = r\cos\Theta \sin\varphi$

$y = r\sin\Theta \sin\varphi$

$z = r\cos\varphi$

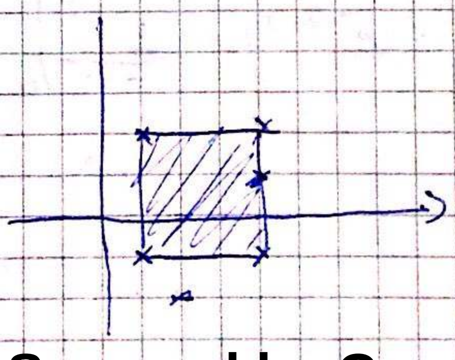
$r = \sqrt{x^2 + y^2 + z^2}$

$\Theta = \operatorname{arctg}\left(\frac{y}{x}\right)$  $\varphi = \cos^{-1}\left(\frac{z}{r}\right)$

Egy négyszögtartomány csúcsai: $A(1;-1)$ $B(4;-1)$ $C(4;2)$ $D(1;2)$

Határozza meg az alábbi integrált

$$\iint_R xy^2 + 3x^2y \, d(x,y)$$



$$\int_1^4 \int_{-1}^2 xy^2 + 3x^2y \, dy\, dx =$$

$$\int_1^4 \left[\frac{xy^3}{3} + 3x^2\frac{y^2}{2}\right]_{-1}^{2} d\cancel{x} = \int_1^4 \left(\frac{2^3 x}{3} + 3x^2 \cdot 2\right) - \left(\frac{x(-1)^3}{3} + 3x^2\frac{(-1)^2}{2}\right) dx$$

$$= \int_1^4 \frac{8}{3}x + 6x^2 + \frac{1}{3}x - \frac{3}{2}x^2\, dx = \int_1^4 3x + \frac{9}{2}x^2\, dx = \left[\frac{3x^2}{2} + \frac{9}{2}\frac{x^3}{3}\right]_1^4$$

$$= \left(\frac{3\cdot 16}{2} + \frac{9\cdot 64}{6}\right) - \left(\frac{3}{2} + \frac{9}{6}\right) = \underline{\underline{117}}$$

Legyen S az origo kp-ú egységgömb

$$\iiint_S 1\, dS = ? \qquad 0 < r < 1$$

$$0 < \theta < 2\pi$$

$$0 < \varphi < \pi$$

$$\int_0^1\int_0^{2\pi}\int_0^{\pi} 1\cdot r^2 \sin\varphi \, d\varphi\, d\theta\, dr = \int_0^1 r^2 dr \cdot \int_0^{2\pi} 1\, d\theta \cdot \int_0^{\pi} \sin\varphi\, d\varphi =$$

$$= \left[\frac{r^3}{3}\right]_0^1 \cdot \left[\theta\right]_0^{2\pi} \cdot \left[-\cos\varphi\right]_0^{\pi} = \frac{1}{3}\cdot 2\pi \cdot 2 = \underline{\underline{\frac{4\pi}{3}}}$$

Adja meg a következő fv-ek Laplace transzformáltját

1,

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$F(s) = \int_0^{\infty} 1 \cdot e^{-st} dt = \left.\frac{e^{-st}}{-s}\right|_0^{\infty} = 0 - \frac{1}{-s} = \underline{\underline{\frac{1}{s}}}$$

2,

$$f(x)=\begin{cases} x & x\geq 0 \\ 0 & x<0 \end{cases}$$

$$F(s)=\mathcal{L}(f)=\int_0^\infty \underset{f}{t}\cdot \underset{g'}{e^{-st}}\,dt = t\cdot\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt = 0-\left[\frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}$$

$$f'=1 \quad g=\frac{e^{-st}}{-s}$$

3,

$$f(x)=\begin{cases} e^{ax} & x\geq 0 \\ 0 & x<0 \end{cases}$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{at}\cdot e^{-st}\,dt = \int_0^\infty e^{at-st}\,dt = \int_0^\infty e^{t(-s+a)}\,dt = \frac{e^{t(-s+a)}}{-s+a}\Big|_0^\infty =$$

$$= 0-\frac{1}{-s+a} = \frac{1}{s-a}$$

Írja fel az $y'+y=0$ DE ált. megoldását

1, Karakterisztikus polinom

$$\lambda^2 + \cancel{\lambda}\,1 = 0$$

$$\lambda^2=-1$$

$$\lambda=\sqrt{-1}=\pm i$$

$$y = A_1\cdot e^{ix} + A_2\cdot e^{-ix}$$

2, ha $y(0)=1$

$$y = A_1 e^{i0} + A_2 e^{-i0} = 1$$

$$A_1 + A_2 = 1 \qquad A_1 = 1 - A_2$$

$$y' = A_1\cdot e^{i0}\cdot i + A_2\cdot e^{-i0}\cdot(-i) = 2$$

$A_1$ [illegible] $(-i) = 2$

$$t_2 = \frac{2-i}{-2i} = \frac{2+i}{2i}$$

3, általános megoldás $y(0)=1 \quad y(\pi)=2$

$$y'' - y = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$$\Rightarrow Y = t_1 e^{x} + t_2 e^{-x}$$

$y(0)=1 \qquad t_1 + t_2 = 1$

$$t_1 = 1 - t_2$$

$y(\pi) = 2 \qquad t_1 e^{\pi} + t_2 e^{-\pi} = 2$

$$(1-t_2)e^{\pi} + t_2 e^{-\pi} = 2$$

$$e^{\pi} - t_2 e^{\pi} + t_2 e^{-\pi} = 2$$

$$t_2(-e^{\pi} + e^{-\pi}) = 2 - e^{\pi}$$

$$t_2 = \frac{2 - e^{\pi}}{-e^{\pi} + e^{-\pi}}$$

Írja fel a következő fv-ek komplex kanonikus alakját

1, $f(z) = \frac{1}{z}$

$$f(x,y) = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} =$$

$$= \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} - i \underbrace{\frac{y}{x^2+y^2}}_{v(x,y)}$$

2, $f(z) = z^2$

$f(x,y) = (x+iy)^2 = x^2 + i2xy + i^2y^2 = x^2 - y^2 + i(2xy)$

Határozza meg az $u(x,y) = e^x \cos y$ harmonikus társát

$u'x = v'y$

$u'y = -v'x$

$u'x = e^x \cdot \cos y \rightarrow \int e^x \cos y \, dx = e^x \cdot \sin y + g(x)$

$u'y = -e^x \sin y \rightarrow \int -e^x \sin y \, dy = e^x \cdot \cos y + h(y)$

$f(z) = e^x \cos y + i(e^x \cdot \sin y)$

Határozza meg a következő vonalintegrált

$a \in \mathbb{C}$ $r$ sugarú kör

$f(z) = (z-a)^n$ $n \in \mathbb{Z}$

$\oint f(z) dz = ?$

$z(t) = a + r \cdot e^{it}$ $z't = r \cdot i \cdot e^{it}$